

Critical independent sets and König–Egerváry graphs

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Abstract

A set S of vertices is *independent* in a graph G , and we write $S \in \text{Ind}(G)$, if no two vertices from S are adjacent, and $\alpha(G)$ is the cardinality of an independent set of maximum size, while $\text{core}(G)$ denotes the intersection of all maximum independent sets [17].

G is called a *König–Egerváry graph* if its order equals $\alpha(G) + \mu(G)$, where $\mu(G)$ denotes the size of a maximum matching. The number $\text{def}(G) = |V(G)| - 2\mu(G)$ is the *deficiency* of G [21].

The number $d(G) = \max\{|S| - |N(S)| : S \in \text{Ind}(G)\}$ is the *critical difference* of G . An independent set A is *critical* if $|A| - |N(A)| = d(G)$, where $N(S)$ is the neighborhood of S , and $\alpha_c(G)$ denotes the maximum size of a critical independent set [26].

In [14] it was shown that G is König–Egerváry graph if and only if there exists a maximum independent set that is also critical, i.e., $\alpha_c(G) = \alpha(G)$.

In this paper we prove that:

(i) $d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = \text{def}(G)$ hold for every König–Egerváry graph G ;

(ii) G is König–Egerváry graph if and only if each maximum independent set of G is critical.

Keywords: independent set, maximum matching, critical difference, critical independent set, deficiency, core.

1 Introduction

Throughout this paper $G = (V, E)$ is a finite, undirected, loopless and without multiple edges graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X . By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. For $F \subset E(G)$, by $G - F$ we denote the partial subgraph of G obtained by deleting the edges of F , and we use $G - e$, if $W = \{e\}$. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, while $N(A) = \cup\{N(v) : v \in A\}$ and $N[A] = A \cup N(A)$ for $A \subset V$.

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent, and by $\text{Ind}(G)$ we mean the set of all the independent sets of G . An independent set of maximum size will be referred to as a *maximum independent set* of G , and the *independence number* of G is $\alpha(G) = \max\{|S| : S \in \text{Ind}(G)\}$.

Let us denote the set $\{S : S \text{ is a maximum independent set of } G\}$ by $\Omega(G)$, and let $\text{core}(G) = \cap\{S : S \in \Omega(G)\}$ [17]. A set $A \subseteq V(G)$ is a *local maximum independent set* of G if $A \in \Omega(G[N[A]])$ [16].

Theorem 1.1 [22] *Every local maximum independent set of a graph is a subset of a maximum independent set.*

A matching (i.e., a set of non-incident edges of G) of maximum cardinality $\mu(G)$ is a *maximum matching*, and a *perfect matching* is one covering all vertices of G .

It is well-known that

$$\lfloor |V|/2 \rfloor + 1 \leq \alpha(G) + \mu(G) \leq |V|$$

hold for any graph $G = (V, E)$. If $\alpha(G) + \mu(G) = |V|$, then G is called a *König-Egerváry graph*. We attribute this definition to Deming [6], and Sterboul [25]. These graphs were studied in [3, 11, 15, 18, 19, 20, 21, 24], and generalized in [2, 23].

According to a well-known result of König [10], and Egerváry [8], any bipartite graph is a König-Egerváry graph. This class includes non-bipartite graphs as well (see, for instance, the graphs H_1 and H_2 in Figure 1).

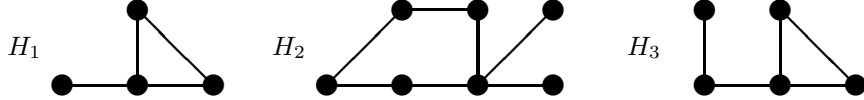


Figure 1: Only H_3 is not a König-Egerváry graph, as $\alpha(H_3) + \mu(H_3) = 4 < 5 = |V(H_3)|$.

It is easy to see that if G is a König-Egerváry graph, then $\alpha(G) \geq \mu(G)$, and that a graph G having a perfect matching is a König-Egerváry graph if and only if $\alpha(G) = \mu(G)$.

The number $d(G) = \max\{|S| - |N(S)| : S \in \text{Ind}(G)\}$ is called the *critical difference* of G . An independent set A is *critical* if $|A| - |N(A)| = d(G)$, and the *critical independence number* $\alpha_c(G)$ is the cardinality of a maximum critical independent set [26]. Clearly, $\alpha_c(G) \leq \alpha(G)$ holds for any graph G . It is known that the problem of finding a critical independent set is polynomially solvable [1, 26].

Proposition 1.2 [13] *If S is a critical independent set, then there is a matching from $N(S)$ into S .*

If S is an independent set of a graph G and $H = G - S$, then we write $G = S * H$. Evidently, any graph admits such representations. For instance, if $E(H) = \emptyset$, then $G = S * H$ is bipartite; if H is complete, then $G = S * H$ is a *split graph* [9].

Proposition 1.3 [18] *G is a König-Egerváry graph if and only if $G = H_1 * H_2$, where $V(H_1) \in \Omega(G)$ and $|V(H_1)| \geq \mu(G) = |V(H_2)|$.*

Let M be a maximum matching of a graph G . To adopt Edmonds's terminology [7], we recall the following terms for G relative to M . An *alternating path* from a vertex x to a vertex y is a x, y -path whose edges are alternating in and not in M . A vertex x is *exposed* relative to M if x is not the endpoint of a heavy edge. An odd cycle C with $V(C) = \{x_0, x_1, \dots, x_{2k}\}$ and $E(C) = \{x_i x_{i+1} : 0 \leq i \leq 2k-1\} \cup \{x_{2k}, x_0\}$, such that $x_1 x_2, x_3 x_4, \dots, x_{2k-1} x_{2k} \in M$ is a *blossom* relative to M . The vertex x_0 is the *base* of the blossom. The *stem* is an even length alternating path joining the base of a blossom and an exposed vertex for M . The base is the only common vertex to the blossom and the stem. A *flower* is a blossom and its stem. A *posy* consists of two (not necessarily disjoint) blossoms joined by an odd length alternating path whose first and last edges belong to M . The endpoints of the path are exactly the bases of the two blossoms. The following result of Sterboul, characterizes König-Egerváry graphs in terms of forbidden configurations.

Theorem 1.4 [25] *For a graph G , the following properties are equivalent:*

- (i) G is a König-Egerváry graph;
- (ii) there exist no flower and no posy relative to some maximum matching M ;
- (iii) there exist no flower and no posy relative to any maximum matching M .

In [20] is given a characterization of König-Egerváry graphs having a perfect matching, in terms of certain forbidden subgraphs with respect to a specific perfect matching of the graph. In [12] is given the following characterization of König-Egerváry graphs in terms of excluded structures.

Theorem 1.5 [12] *Let M be a maximum matching in a graph G . Then G is a König-Egerváry graph if and only if G does not contain one of the forbidden configurations, depicted in Figure 2, with respect to M .*

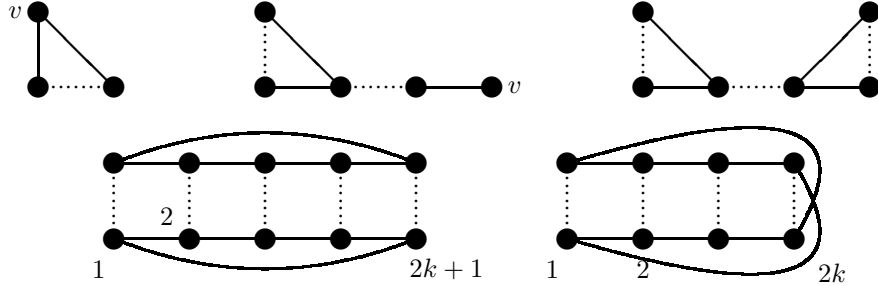


Figure 2: Forbidden configurations. The vertex v is not adjacent to the matching edges (namely, dashed edges).

In [14] it was shown that G is a König-Egerváry graph if and only if $\alpha_c(G) = \alpha(G)$, thus giving a positive answer to the Graffiti.pc 329 conjecture [5].

The *deficiency* of G , denoted by $def(G)$, is defined as the number of exposed vertices relative to a maximum matching [21]. In other words, $def(G) = |V(G)| - 2\mu(G)$.

In this paper we prove that the critical difference for a König-Egerváry graph G is given by

$$d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = \text{def}(G),$$

and using this finding, we show that G is a König-Egerváry graph if and only if each of its maximum independent sets is critical.

2 Results

Proposition 2.1 *Every critical independent set is a local maximum independent set.*

Proof. Suppose, on the contrary, that there is a critical independent set S such that $S \notin \Psi(G)$, i.e., there exists some independent set $A \subseteq N[S]$, larger than S . It follows that $|A \cap N(S)| > |S - S \cap A|$, and this contradicts the fact that, according to Proposition 1.2, there is a matching from $A \cap N(S)$ to S , in fact, from $A \cap N(S)$ to $S - S \cap A$. ■

The converse of Proposition 2.1 is not true; e.g., the set $\{d, h\}$ is a local maximum independent set of the graph G_1 from Figure 3, but it is not critical.

Using Theorem 1.1, we easily deduce the following result.

Corollary 2.2 [4] *Every critical independent set is contained in some maximum independent set.*

Theorem 2.3 *If G is a König-Egerváry graph, then*

- (i) [18] $N(\text{core}(G)) = \cap \{V(G) - S : S \in \Omega(G)\};$
- (ii) [19] $\alpha(G) + |\cap \{V(G) - S : S \in \Omega(G)\}| = \mu(G) + |\cap \{S : S \in \Omega(G)\}|;$
- (iii) [19] $G - N[\text{core}(G)]$ has a perfect matching and it is also a König-Egerváry graph.

Let us notice that for non-König-Egerváry graphs every relation between $\alpha(G) - \mu(G)$ and $|\text{core}(G)| - |N(\text{core}(G))|$ is possible.

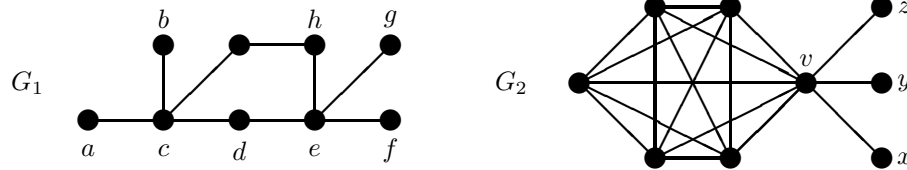


Figure 3: $\alpha(G_1) = 6$, $\mu(G_1) = 3$, $\text{core}(G_1) = \{a, b, d, g, f\}$ and $N(\text{core}(G_1)) = \{c, e\}$, while $\alpha(G_2) = 4$, $\mu(G_2) = 3$, $\text{core}(G_2) = \{x, y, z\}$, and $N(\text{core}(G_2)) = \{v\}$.

The non-König-Egerváry graphs from Figure 3 satisfy:

$$\alpha(G_1) - \mu(G_1) = 3 = |\text{core}(G_1)| - |N(\text{core}(G_1))|$$

and

$$\alpha(G_2) - \mu(G_2) = 1 < 2 = |\text{core}(G_2)| - |N(\text{core}(G_2))|.$$

The opposite direction of the above inequality may be found in $G_3 = K_{2n} - e$, $n \geq 3$:

$$\alpha(G_3) - \mu(G_3) = 2 - n > 4 - 2n = 2 - (2n - 2) = |\text{core}(G_3)| - |N(\text{core}(G_3))|.$$

Theorem 2.4 *If G is König-Egerváry graph, then the following equalities hold*

$$d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = \text{def}(G).$$

Proof. Firstly, let us prove that $\alpha(G) - \mu(G) \geq |S| - |N(S)|$ holds for every $S \in \text{Ind}(G)$, i.e., $d(G) \leq \alpha(G) - \mu(G)$. If $\alpha(G) = \mu(G)$, then G has a perfect matching and

$$|S| - |N(S)| \leq 0 = \alpha(G) - \mu(G)$$

holds for every $S \in \text{Ind}(G)$.

Suppose that $\alpha(G) > \mu(G)$. Let $S_0 \in \Omega(G)$ and M be a maximum matching, i.e., $|M| = |V(G) - S_0| = \mu(G)$. Assume that $S \in \text{Ind}(G)$ satisfies $|S| - |N(S)| > 0$. Then one can write $S = S_1 \cup S_2 \cup S_3$, where $S_3 \subseteq V(G) - S_0$, $S_1 \cup S_2 \subset S_0$, $S_1 \cap S_2 = \emptyset$, and S_2 contains every $v \in S$ matched by M with some vertex of $V(G) - S_0$. Since M is a maximum matching, we obtain that $|S_2| - |N(S_2)| \leq 0$ and $|S_3| - |N(S_3)| \leq 0$. Consequently, we infer that

$$\alpha(G) - \mu(G) = |S_0| - |V(G) - S_0| \geq |S_1| \geq |S| - |N(S)|,$$

as required (see Figure 4 for various examples of S).

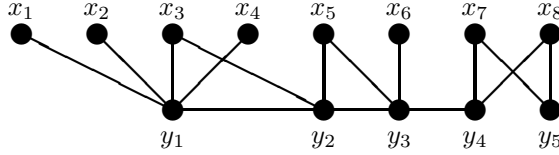


Figure 4: $S_0 = \{x_i : 1 \leq i \leq 8\}$, $M = \{y_1x_4, y_2x_5, y_3x_6, y_4x_7, y_5x_8\}$, $S = S_1 \cup S_2 \cup S_3$, where $S_2 = \{x_5\}$, $S_3 = \{y_4, y_5\}$, while S_1 belongs to $\{\{x_1, x_2\}, \{x_1x_3\}, \{x_3\}\}$.

The fact that $\text{core}(G)$ is an independent set of G ensures that

$$\alpha(G) - \mu(G) \geq |\text{core}(G)| - |N(\text{core}(G))|.$$

Since G is a König-Egerváry graph, we get that

$$\alpha(G) + \mu(G) = |V(G)| = |\text{core}(G)| + |N(\text{core}(G))| + |V(G - N[\text{core}(G)])|.$$

Assuming that

$$\alpha(G) - \mu(G) > |\text{core}(G)| - |N(\text{core}(G))|,$$

we obtain the following contradiction

$$\begin{aligned} 2\alpha(G) &> 2|\text{core}(G)| + |V(G - N[\text{core}(G)])| \\ &= 2|\text{core}(G)| + 2\alpha(G - N[\text{core}(G)]) = 2\alpha(G), \end{aligned}$$

because $|V(G - N[\text{core}(G)])| = 2\alpha(G - N[\text{core}(G)])$ by Theorem 2.3(iii).

Therefore, we get that $\alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))|$. Actually, this equality immediately follows from Theorem 2.3(i),(ii), but the current way of proof exploits different aspects of $\text{Ind}(G)$.

Further, using the inequality $d(G) \leq \alpha(G) - \mu(G)$ and the equality

$$\alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))|,$$

we finally deduce that

$$\begin{aligned} |\text{core}(G)| - |N(\text{core}(G))| &\leq \max\{|S| - |N(S)| : S \in \text{Ind}(G)\} = d(G) \\ &\leq \alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))|, \end{aligned}$$

i.e.,

$$\alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))| = d(G).$$

Since G is a König-Egerváry graph, we infer that

$$\alpha(G) - \mu(G) = \alpha(G) + \mu(G) - 2\mu(G) = |V(G)| - 2\mu(G) = \text{def}(G),$$

and this completes the proof. ■

Corollary 2.5 *If G is a König-Egerváry graph, then $d(G) = 0$ if and only if G has a perfect matching.*

Remark 2.6 *There exist non-König-Egerváry graphs enjoying the equalities*

$$d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G),$$

see, for instance, the graph G from Figure 5.

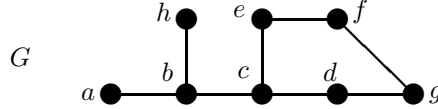


Figure 5: G has $\alpha(G) = 4$, $\mu(G) = 3$, $\text{core}(G) = \{a, h\}$ and $N(\text{core}(G)) = \{b\}$.

Theorem 2.7 *The following assertions are equivalent:*

- (i) G is a König-Egerváry graph;
- (ii) there is $S \in \Omega(G)$, such that S is critical, i.e., $\alpha_c(G) = \alpha(G)$;
- (iii) every $S \in \Omega(G)$ is critical.

Proof. (i) \implies (iii) Let $S \in \Omega(G)$, $A = S - \text{core}(G)$ and $B = V(G) - S - N(\text{core}(G))$. By Theorem 2.3(iii), we infer that $|A| = |B|$, since $G - N[\text{core}(G)]$ has a perfect matching. Hence, we obtain that

$$\begin{aligned} |S| - |N(S)| &= |A| + |\text{core}(G)| - (|B| + |N(\text{core}(G))|) \\ &= |\text{core}(G)| - |N(\text{core}(G))|. \end{aligned}$$

In other words, according to Theorem 2.4, the equality $|S| - |N(S)| = d(G)$ is true for every $S \in \Omega(G)$.

(iii) \implies (ii) It is clear.

(ii) \implies (i) This was done in [14]. For the sake of completeness we add the proof.

There is a critical independent set S with $|S| = \alpha_c(G) = \alpha(G)$. By Proposition 1.2, there exists a matching M from $N(S)$ into S , and clearly, $|M| = |N(S)| = \mu(G)$. Hence, we finally obtain that $|V(G)| = |S| + |N(S)| = \alpha(G) + \mu(G)$, i.e., G is a König-Egerváry graph. ■

3 Conclusions

In this paper we give a new characterization of König-Egerváry graphs. On the one hand, it is similar in form to Sterboul's theorem [25]. On the other hand it extends Larson's finding [14]. We found that the critical difference of a König-Egerváry graph G is given by

$$d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = \text{def}(G).$$

It seems interesting to find other families of graphs satisfying these equalities.

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